Theoretical Question 1

A Swing with a Falling Weight

A rigid cylindrical rod of radius R is held horizontal above the ground. With a string of negligible mass and length L ($L > 2\pi R$), a pendulum bob of mass m is suspended from point A at the top of the rod as shown in Figure 1a. The bob is raised until it is level with A and then released from rest when the string is taut. Neglect any stretching of the string. Assume the pendulum bob may be treated as a mass point and swings only in a plane perpendicular to the axis of the rod. Accordingly, the pendulum bob is also referred to as the *particle*. The acceleration of gravity is $\vec{g}$.

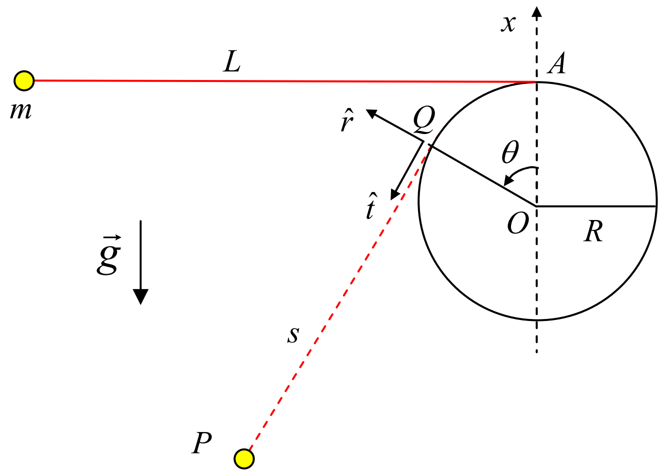


Figure 1a

Let O be the origin of the coordinate system. When the particle is at point P, the string is tangential to the cylindrical surface at Q. The length of the line segment QP is called s. The unit tangent vector and the unit radial vector at Q are given by $\hat{t}$ and $\hat{r}$, respectively. The angular displacement θ of the radius OQ, as measured counterclockwise from the vertical x-axis along OA, is taken to be positive.

When $\theta = 0$, the length s is equal to L and the gravitational potential energy U of the particle is zero. As the particle moves, the instantaneous time rates of change of θ and s are given by $\dot{\theta}$ and $\dot{s}$, respectively.

Unless otherwise stated, all the speeds and velocities are relative to the fixed point O.

Part A

In Part A, the string is taut as the particle moves. In terms of the quantities introduced above (i.e., s, θ, $\dot{s}$, $\dot{\theta}$, R, L, g, $\hat{t}$ and $\hat{r}$), find:

(a) The relation between $\dot{\theta}$ and $\dot{s}$. [0.5 point]

(b) The velocity $\vec{v}_Q$ of the moving point Q relative to O. [0.5 point]

(c) The particle's velocity $\vec{v}\,'$ relative to the moving point Q when it is at P. [0.7 point]

(d) The particle's velocity $\vec{v}$ relative to O when it is at P. [0.7 point]

(e) The $\hat{t}$ -component of the particle's *acceleration* relative to O when it is at P. [0.7 point]

(f) The particle's gravitational potential energy U when it is at P. [0.5 point]

(g) The speed v_m of the particle at the lowest point of its trajectory. [0.7 point]

Part B

In Part B, the ratio L to R has the following value:

$$\frac{L}{R}=\frac{9\pi}{8}+\frac{2}{3}\cot\frac{\pi}{16}=3.534+3.352=6.886$$

(h) What is the speed v_s of the particle when the string segment from Q to P is both straight and shortest in length? (in terms of g and R) [2.4 points]

(i) What is the speed v_H of the particle at its highest point H when it has swung to the other side of the rod? (in terms of g and R [1.9 points]

Part C

In Part C, instead of being suspended from A, the pendulum bob of mass m is connected by a string over the top of the rod to a heavier weight of mass M, as shown in Figure 1b. The weight can also be treated as a particle.

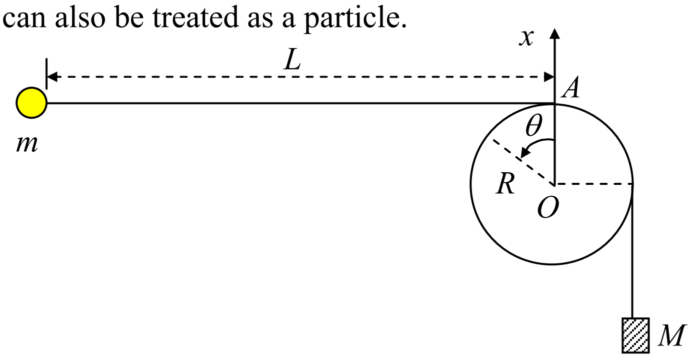


Figure 1b

Initially, the bob is held stationary at the same level as A so that, with the weight hanging below O, the string is taut with a horizontal section of length L. The bob is then released from rest and the weight starts falling. Assume that the bob remains in a vertical plane and can swing past the falling weight without any interruption.

The kinetic friction between the string and the rod surface is negligible. But the static friction is assumed to be large enough so that the weight will remain stationary once it has come to a stop (i.e. zero velocity).

(j) Assume that the weight indeed comes to a stop after falling a distance D and that $(L\text{-}D) >> R$. If the particle can then swing around the rod to $\theta = 2\pi$ while both segments of the string free from the rod remain straight, the ratio $\alpha = D/L$ must not be smaller than a critical value α_c. Neglecting terms of the order R/L or higher, obtain an estimate on α_c in terms of M/m. [3.4 points]

*[**Answer Sheet**]* *Theoretical Question 1*

A Swing with a Falling Weight

(a) The relation between $\dot{\theta}$ and $\dot{s}$ is

(b) The velocity of the moving point Q relative to O is

$\vec{v}_Q =$

(c) When at P, the particle's velocity relative to the moving point Q is

$\vec{v}' =$

(d) When at P, the particle's velocity relative to O is

$\vec{v} =$

(e) When at P, the $\hat{t}$-component of the particle's acceleration relative to O is

(f) When at P, the particle's gravitational potential energy is

$U =$

(g) The particle's speed when at the lowest point of its trajectory is

$v_m =$

(h) When line segment QP is straight with the shortest length, the particle's speed is (Give *expression and value in terms of g and R*)

$v_s =$

(i) At the highest point, the particle's speed is (Give *expression and value in terms of g and R*)

$v_H =$

(j) In terms of the mass ratio M/m, the critical value α_c of the ratio D/L is

$\alpha_c =$

Theoretical Question 2

A Piezoelectric Crystal Resonator under an Alternating Voltage

Consider a uniform rod of unstressed length ℓ and cross-sectional area A (Figure 2a). Its length changes by $\Delta\ell$ when equal and opposite forces of magnitude F are applied to its ends faces normally. The *stress* T on the end faces is defined to be F/A. The *fractional change* in its length, i.e., $\Delta\ell/\ell$, is called the *strain* S of the rod. In terms of stress and strain, Hooke's law may be expressed as

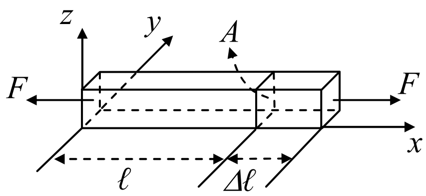
$$T = Y\,S \quad \text{or} \quad \frac{F}{A} = Y\frac{\Delta\ell}{\ell} \tag{1}$$

where Y is called the *Young's modulus* of the rod material. Note that a *compressive* stress T corresponds to $F < 0$ and a decrease in length (i.e., $\Delta\ell < 0$). Such a stress is thus negative in value and is related to the pressure p by $T = -p$.

For a uniform rod of density ρ, the speed of propagation of longitudinal waves (i.e., sound speed) along the rod is given by

$$u = \sqrt{Y/\rho} \tag{2}$$

Figure 2a



The effect of damping and dissipation can be ignored in answering the following questions.

Part A: mechanical properties

A uniform rod of semi-infinite length, extending from $x = 0$ to ∞ (see Figure 2b), has a density ρ. It is initially stationary and unstressed. A piston then steadily exerts a small pressure p on its left face at $x = 0$ for a very short time Δt, causing a pressure wave to propagate with *speed* u to the right.

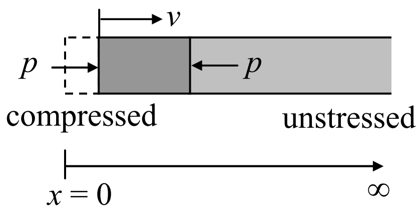


Figure 2b

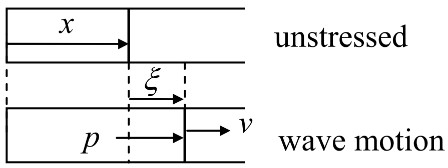


Figure 2c

(a) If the piston causes the rod's left face to move at a constant velocity v (Figure 2b), what are the strain S and pressure p at the left face during the time Δt? *Answers must be given in terms of ρ, u, and v only.* [1.6 points]

(b) Consider a longitudinal wave traveling along the x direction in the rod. For a cross section at x when the rod is unstressed (Figure 2c), let $\xi(x, t)$ be its displacement at time t and assume

$$\xi(x,t)=\xi_0 \sin k(x-ut) \qquad (3)$$

where ξ_0 and k are constants. Determine the corresponding velocity $v(x, t)$, strain $S(x, t)$, and pressure $p(x, t)$ as a function of x and t. [2.4 points]

Part B: electromechanical properties (including piezoelectric effect)

Consider a quartz crystal slab of length b, thickness h, and width w (Figure 2d). Its length and thickness are along the x-axis and z-axis. Electrodes are formed by thin metallic coatings at its top and bottom surfaces. Electrical leads that also serve as mounting support (Figure 2e) are soldered to the electrode's centers, which may be assumed to be stationary for longitudinal oscillations along the x direction.

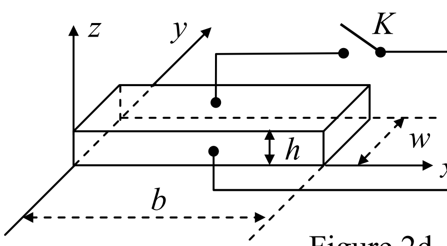


Figure 2d

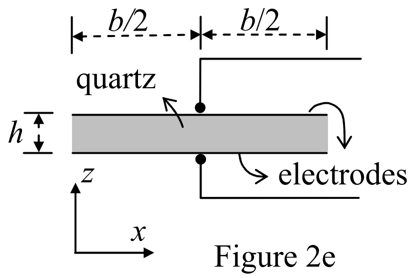


Figure 2e

The quartz crystal under consideration has a density ρ of 2.65×10^3 kg/m^3 and Young's modulus Y of 7.87×10^{10} N/m^2. The length b of the slab is 1.00 cm and the width w and height h of the slab are such that $h \ll w$ and $w \ll b$. With switch K left open, we assume only *longitudinal modes* of standing wave oscillation in the x direction are excited in the quartz slab.

For a standing wave of frequency $f=\omega/2\pi$, the displacement $\xi(x, t)$ at time t of a cross section of the slab with equilibrium position x may be written as

$$\xi(x,t)=2\xi_0\, g(x)\cos\omega t, \qquad (0\le x\le b) \qquad (4a)$$

where ξ_0 is a positive constant and the spatial function $g(x)$ is of the form

$$g(x)=B_1\sin k(x-\frac{b}{2})+B_2\cos k(x-\frac{b}{2}). \qquad (4b)$$

g(x) has the maximum value of one and $k=\omega/u$. Keep in mind that the centers of the electrodes are stationary and the left and right faces of the slab are free and must have zero stress (or pressure).

(c) Determine the values of B_1 and B_2 in Eq. (4b) for a longitudinal standing wave in the quartz slab. [1.2 point]

(d) What are the two lowest frequencies at which longitudinal standing waves may be excited in the quartz slab? [1.2 point]

The *piezoelectric* effect is a special property of a *quartz* crystal. Compression or dilatation of the crystal generates an electric voltage across the crystal, and conversely, an external voltage applied across the crystal causes the crystal to expand or contract depending on the polarity of the voltage. Therefore, mechanical and electrical oscillations can be coupled and made to resonate through a quartz crystal.

To account for the piezoelectric effect, let the surface charge densities on the upper and lower electrodes be $-\sigma$ and $+\sigma$, respectively, when the quartz slab is under an electric field E in the z direction. Denote the slab's strain and stress in the x direction by S and T, respectively. Then the piezoelectric effect of the quartz crystal can be described by the following set of equations:

$$S = (1/Y)T + d_p E \tag{5a}$$

$$\sigma = d_p T + \varepsilon_T E \tag{5b}$$

where $1/Y = 1.27\times10^{-11}$ m^2/N is the *elastic compliance* (i.e., inverse of Young's modulus) at constant electric field and $\varepsilon_T = 4.06\times10^{-11}$ F/m is the *permittivity* at constant stress, while $d_p = 2.25\times10^{-12}$ m/V is the *piezoelectric coefficient*.

Let switch K in Fig. 2d be closed. The alternating voltage $V(t) = V_m \cos \omega t$ now acts across the electrodes and a *uniform* electric field $E(t) = V(t)/h$ in the z direction appears in the quartz slab. When a steady state is reached, a longitudinal standing wave oscillation of angular frequency ω appears in the slab in the x direction.

With E being uniform, the wavelength λ and the frequency f of a longitudinal standing wave in the slab are still related by $\lambda = u/f$ with u given by Eq. (2). But, as Eq. (5a) shows, $T = YS$ is no longer valid, although the definitions of strain and stress remain unchanged and the end faces of the slab remain free with zero stress.

(e) Taking Eqs. (5a) and (5b) into account, the surface charge density σ on the lower electrode as a function of x and t is of the form,

$$\sigma(x,t) = \left[D_1 \cos k\left(x - \frac{b}{2}\right) + D_2\right]\frac{V(t)}{h},$$

where $k = \omega/u$. Find the expressions for D_1 and D_2. [2.2 points]

(f) The total surface charge $Q(t)$ on the lower electrode is related to $V(t)$ by

$$Q(t) = [1 + \alpha^2(\frac{2}{kb}\tan\frac{kb}{2} - 1)]C_0 V(t) \tag{6}$$

Find the expression for C_0 and the expression and numerical value of α^2.

[1.4 points]

*[**Answer Sheet**]* *Theoretical Question 2*

A Piezoelectric Crystal Resonator under an Alternating Voltage

***Wherever requested*, give *each answer* as *analytical expressions* followed by *numerical values* and *units*. *For example*:** area of a circle $A = \pi r^2 = 1.23\ \text{m}^2$.

(a) The strain S and pressure p at the left face are (*in terms of* ρ, u, *and* v)

$S =$
$p =$

(b) The velocity $v(x, t)$, strain $S(x, t)$, and pressure $p(x, t)$ are

$v(x, t) =$
$S(x, t) =$
$p(x, t) =$

(c) The values of B_1 and B_2 are

$B_1 =$
$B_2 =$

(d) The lowest two frequencies of standing waves are (*expression and value*)

The Lowest
The Second Lowest

(e) The expressions of D_1 and D_2 are

$D_1 =$
$D_2 =$

(f) The constants α^2 (*expression and value*) and C_0 are (*expression only*)

$\alpha^2 =$
$C_0 =$

Theoretical Question 3

Part A

Neutrino Mass and Neutron Decay

A free neutron of mass m_n decays at rest in the laboratory frame of reference into three non-interacting particles: a proton, an electron, and an anti-neutrino. The rest mass of the proton is m_p, while the rest mass of the anti-neutrino m_ν is assumed to be nonzero and much smaller than the rest mass of the electron m_e. Denote the speed of light in vacuum by c. The measured values of mass are as follows:

$$m_n = 939.56563 \text{ MeV}/c^2,\ m_p = 938.27231 \text{ MeV}/c^2,\ m_e = 0.5109907 \text{ MeV}/c^2$$

In the following, all energies and velocities are referred to the laboratory frame. Let E be the total energy of the electron coming out of the decay.

(a) Find the maximum possible value $E_{\max}$ of E and the speed v_{m} of the anti-neutrino when $E = E_{\max}$. Both answers must be expressed in terms of the rest masses of the particles and the speed of light. Given that $m_\nu < 7.3 \text{ eV}/c^2$, compute $E_{\max}$ and the ratio v_{m}/c to 3 significant digits. [4.0 points]

Part B

Light Levitation

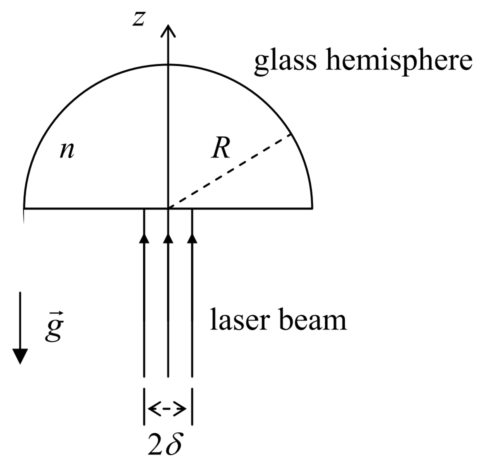
A transparent glass hemisphere with radius R and mass m has an index of refraction n. In the medium outside the hemisphere, the index of refraction is equal to one. A parallel beam of monochromatic laser light is incident uniformly and normally onto the central portion of its planar surface, as shown in Figure 3a. The acceleration of gravity $\vec{g}$ is vertically downwards. The radius δ of the circular cross-section of the laser beam is much smaller than R. Both the glass hemisphere and the laser beam are axially symmetric with respect to the z-axis.

The glass hemisphere does not absorb any laser light. Its surface has been coated with a thin layer of transparent material so that reflections are negligible when light enters and leaves the glass hemisphere. The optical path traversed by laser light passing through the non-reflecting surface layer is also negligible.

(b) Neglecting terms of the order $(\delta/R)^3$ or higher, find the laser power P needed to balance the weight of the glass hemisphere. [4.0 points]

Hint: $\cos\theta \approx 1-\theta^2/2$ when θ is much smaller than one.

Figure 3a



*[**Answer Sheet**]* *Theoretical Question 3*

***Wherever requested*, give *each answer* as *analytical expressions* followed by *numerical values* and *units*. *For example*:** area of a circle $A = \pi r^2 = 1.23\ \text{m}^2$.

Neutrino Mass and Neutron Decay

(a) (Give expressions in terms of rest masses of the particles and the speed of light)

The maximum energy of the electron is (*expression and value*)

$E_{max} =$

The ratio of anti-neutrino's speed at $E = E_{max}$ to c is (*expression and value*)

$v_m / c =$

Light Levitation

(b) The laser power needed to balance the weight of the glass hemisphere is

$P =$